

AN ANALOGUE OF BISHOP'S APPROXIMATION IN  $\mathbb{C}^n$ 

SÖNMEZ ŞAHUTOĞLU AND AKAKI TIKARADZE

**ABSTRACT.** We prove an analogue of Bishop's approximation theorem on a class of domains in  $\mathbb{C}^n$  on which the  $\bar{\partial}$ -problem is solvable with  $L^\infty$  estimates. Furthermore, as a corollary we obtain a version of Axler-Čučković-Rao Theorem in higher dimensions.

Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $\phi$  be a complex-valued function on  $\Omega$ . Let  $H^\infty(\Omega)$  and  $H^\infty(\Omega)[\phi]$  denote the set of bounded holomorphic functions on  $\Omega$  and the algebra generated by  $\phi$  over  $H^\infty(\Omega)$ , respectively. The main goal of this paper is to give a higher dimensional version of the following approximation result by Bishop [Bis89, Theorem 1.2].

**Theorem (Bishop).** *Let  $\Omega$  be an open set in  $\mathbb{C}$  and  $f$  be a bounded holomorphic function on  $\Omega$  that is non-constant on every connected component of  $\Omega$ . Then  $H^\infty(\Omega)[\bar{f}]$  is dense in  $C(\bar{\Omega})$  in the uniform topology.*

We remark that for a more restrictive class of domains Bishop also proved a stronger approximation result [Bis89, Theorem 1.1] where  $\bar{f}$  is only assumed to be a non-holomorphic harmonic function. Such a result for the unit disc goes back to Axler and Shields [AS87].

Recently, Cao gave an incorrect statement [Cao08, Theorem 5] in an attempt to give a higher dimensional version of Bishop's Theorem (Izzo-Li [IL13, pg 246] noticed that the statement is incorrect). This article is motivated by these papers and it is an attempt to contribute version of Bishop's Theorem on domains in  $\mathbb{C}^n$ .

Let  $\Omega \subset \mathbb{C}^n$  be a pseudoconvex domain and  $CL_{(0,q)}^\infty(\Omega)$  denote the set of  $(0,q)$ -forms with coefficient functions that are  $C^\infty$ -smooth and bounded on  $\Omega$ . That is,  $CL_{(0,q)}^\infty(\Omega) = L_{(0,q)}^\infty(\Omega) \cap C_{(0,q)}^\infty(\Omega)$ . We call  $\Omega$  a  $L^\infty$ -pseudoconvex domain if for  $1 \leq q \leq n$ , and  $f \in CL_{(0,q)}^\infty(\Omega)$  such that  $\bar{\partial}f = 0$  there exists  $g \in L_{(0,q-1)}^\infty(\Omega)$  such that  $\bar{\partial}g = f$ .

The class of  $L^\infty$ -pseudoconvex domains include the products of  $C^2$ -smooth bounded strongly pseudoconvex domains [SH80], smooth bounded pseudoconvex finite type domains in  $\mathbb{C}^2$  [Ran90], smooth bounded finite type convex domains in  $\mathbb{C}^n$  [DFF99], and some infinite type smooth bounded convex domains in  $\mathbb{C}^2$  [FLZ11].

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Before stating our results it will be convenient to introduce some notation. Given a holomorphic mapping  $f : \Omega \rightarrow \mathbb{C}^m$  (where  $\Omega \subset \mathbb{C}^n$ ) and  $\lambda \in \mathbb{C}^m$  we denote the union of all non-isolated points of  $f^{-1}(\lambda)$  by  $\Omega_{f,\lambda}$ . Since  $f^{-1}(\lambda)$  is a complex subvariety of  $\Omega$  (for  $\lambda$  in the range of  $f$ ), it follows that  $\Omega_{f,\lambda}$  is the union of all positive dimensional connected components of  $f^{-1}(\lambda)$ . In the case  $f$  extends smoothly up to the boundary of  $\Omega$ , we define  $\Omega'_{f,\lambda}$  to be the union of all non-isolated points of  $f^{-1}(\lambda)$  within  $\overline{\Omega}$ . Clearly  $\Omega'_{f,\lambda} \subset \Omega_{f,\lambda} \cup b\Omega$  where  $b\Omega$  denotes the boundary of  $\Omega$ . We define

$$\Omega_f = \bigcup_{\lambda \in \mathbb{C}^m} \Omega_{f,\lambda}.$$

It is clear that  $\Omega_f$  is a subset of the set where the Jacobian of  $f$  has rank strictly less than  $n$ .

The main results of this paper are the following theorems.

**Theorem 1.** *Let  $\Omega$  be a bounded  $L^\infty$ -pseudoconvex domain in  $\mathbb{C}^n$  and  $f_j \in H^\infty(\Omega)$  for  $j = 1, \dots, m$ . Assume that  $g \in C(\overline{\Omega})$  such that  $g|_{b\Omega \cup \Omega_f} = 0$  where  $f = (f_1, \dots, f_m)$ . Then  $g$  belongs to the closure of  $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$  in  $L^\infty(\Omega)$ .*

To formulate our next result we will need the following notation. The set of holomorphic functions on  $\Omega$  that have smooth extension up to the boundary is denoted by  $A^\infty(\Omega)$ . Given a compact set  $K \subset \overline{\Omega}$ , we will denote by  $A_{\overline{\Omega}}(K)$  the norm closed subalgebra of continuous functions on  $K$  spanned by restrictions of  $A^\infty(U \cap \Omega)$  on  $K$ , where  $U$  runs through open neighborhoods of  $K$ .

**Theorem 2.** *Let  $\Omega$  be a smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  and  $f_j \in A^\infty(\Omega)$  for  $j = 1, \dots, m$ . Then  $g \in C(\overline{\Omega})$  belongs to the closure of  $A^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$  in  $L^\infty(\Omega)$  if and only if for any  $\lambda$  in the range of  $f = (f_1, \dots, f_m)$  we have  $g|_{\Omega'_{f,\lambda}} \in A_{\overline{\Omega}}(\Omega'_{f,\lambda})$ .*

*Remark 1.* Let us explain the relation between our results and recent results of Izzo [Izz11] and Samuelsson-Wold [SW12].

As pointed out to us by Alex Izzo, weaker versions of our results can be obtained from [Izz11]. For example, if  $\Omega_f$  in Theorem 1 is replaced by the set of points where  $J_f$ , the Jacobian of  $f$ , has rank strictly less than  $n$  (usually a much larger set than  $\Omega_f$ ), then [Izz11, Theorem 1.3] implies the conclusion of Theorem 1 when  $\Omega$  is strongly pseudoconvex domain. This can be seen by applying [Izz11, Theorem 1.3] to the uniform algebra on  $X$ , the maximal spectrum of  $H^\infty(\Omega)$ , generated by  $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$ .

On the other hand, Samuelsson-Wold's result [SW12, Theorem 1.3] requires the assumption that  $\Omega_f$  is an empty set and  $\Omega$  belongs to a smaller class of domains with less regular boundary. We also remark that the results of Izzo and Samuelsson-Wold do not require that  $f_j$ 's are smooth up to the boundary.

In case the rank of  $J_f$  is  $n$  for some  $z \in \Omega$ , the set of points at which  $J_f$  has rank strictly less than  $n$  is a closed set of measure 0 [Ran86, Theorem 3.7]. Furthermore, one can show that the set of functions  $g \in C_0^\infty(\Omega)$  that vanish where  $J_f$  has rank strictly less than  $n$ , are dense in  $L^p(\Omega)$  for  $0 < p < \infty$ . Therefore, Theorem 1 leads to the following corollary.

**Corollary 1.** *Let  $\Omega$  be a bounded  $L^\infty$ -pseudoconvex domain in  $\mathbb{C}^n$  and  $f_j \in H^\infty(\Omega)$  for  $j = 1, \dots, m$  and  $n \leq m$ . Then  $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$  is dense in  $L^p(\Omega)$  for all  $0 < p < \infty$  if and only if the Jacobian of  $f$  has rank  $n$  for some  $z \in \Omega$ .*

We will remark that the above result is optimal, in the sense that by a result of Izzo and Li [IL13, Theorem 4.2], in order for  $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$  to be dense in  $L^p(\Omega)$ , it is necessary that the Jacobian of  $f_1, \dots, f_m$  has rank  $n$  outside of a set with empty interior (and of measure 0).

As an application to operator theory on Bergman spaces, we get the following corollary about commutants of Toeplitz operators with holomorphic symbols, generalizing a well-known result by Axler-Čučković-Rao [AČR00] to higher dimensions. Let  $A^2(\Omega)$  denote the space of square integrable holomorphic functions on  $\Omega$  and  $P : L^2(\Omega) \rightarrow A^2(\Omega)$  be the Bergman projection, the orthogonal projection onto  $A^2(\Omega)$ . For  $g \in L^\infty(\Omega)$ , the Toeplitz operator  $T_g : A^2(\Omega) \rightarrow A^2(\Omega)$  is defined as  $T_g f = P(gf)$  for all  $f \in A^2(\Omega)$ .

**Corollary 2.** *Let  $\Omega$  be a bounded  $L^\infty$ -pseudoconvex domain in  $\mathbb{C}^n$ ,  $g \in L^\infty(\Omega)$ , and  $f_j \in H^\infty(\Omega)$  for  $j = 1, \dots, m$  and  $n \leq m$ . Assume that the Jacobian of the function  $f = (f_1, \dots, f_m) : \Omega \rightarrow \mathbb{C}^m$  has rank  $n$  for some  $z \in \Omega$  and  $T_g$  commutes with  $T_{f_j}$  for  $1 \leq j \leq m$ . Then  $g$  is holomorphic.*

This paper is organized as follows: The next section contains relevant basic facts and results about  $\bar{\partial}$ -Koszul complex. Then we will present the proofs of Theorems 1 and 2. We will finish the paper with the proof of Corollary 2.

## THE $\bar{\partial}$ -KOSZUL COMPLEX

Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $V$  be a vector space of dimension  $m$  with a basis  $\{e_1, e_2, \dots, e_m\}$ . We define

$$\wedge^r V = \{e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_r} : j_1 < j_2 < \dots < j_r\}$$

and  $\Gamma_{(r,s)}^\infty = \wedge^r V \otimes CL_{(0,s)}^\infty(\Omega)$  where  $r$  and  $s$  are nonnegative integers. We note that throughout the paper we use the convention that  $\Gamma_{(r,s)}^\infty = \{0\}$  if  $r \geq m+1$  or  $s \geq n+1$ . Finally,  $CL_{(0,0)}^\infty(\Omega) = CL^\infty(\Omega)$ .

We define the unbounded operator  $\bar{\partial} : \Gamma_{(r,s)}^\infty \rightarrow \Gamma_{(r,s+1)}^\infty$  as  $\bar{\partial}(e_J \otimes W) = e_J \otimes \bar{\partial}W$  where  $e_J \in \wedge^r V$  and  $W \in CL_{(0,s)}^\infty(\Omega)$ . The operator  $\bar{\partial}$  is defined on

$$Dom_\infty(\bar{\partial}) = \left\{ f \in \Gamma_{(r,s)}^\infty : \bar{\partial}f \in \Gamma_{(r,s+1)}^\infty \right\}.$$

Let  $f = (f_1, \dots, f_m) : \Omega \rightarrow \mathbb{C}^m$  be a bounded holomorphic mapping. Then for  $0 \leq s \leq n$  and  $0 \leq r \leq m$  we define the operator

$$\mathcal{T}_f : \Gamma_{(r+1,s)}^\infty \rightarrow \Gamma_{(r,s)}^\infty$$

with the following properties:

- (1)  $\mathcal{T}_f(e_j \otimes W) = f_j W$ ,
- (2)  $\mathcal{T}_f(A \wedge B) = \mathcal{T}_f(A) \wedge B + (-1)^{|A|_1} A \wedge \mathcal{T}_f B$  (here  $|\cdot|_1$  is the order of  $A$  in  $\cup_{r=0}^m \Lambda^r V$ ),
- (3)  $\mathcal{T}_f \bar{\partial} = \bar{\partial} \mathcal{T}_f$  on  $Dom_\infty(\bar{\partial})$  for  $0 \leq s \leq n$  and  $0 \leq r \leq m$ ,
- (4)  $\mathcal{T}_f \mathcal{T}_f = 0$  and  $\bar{\partial} \bar{\partial} = 0$ .

We note that  $\mathcal{T}_f W = 0$  for  $W \in \Gamma_{(0,s)}^\infty$  and  $0 \leq s \leq n$ .

**Lemma 1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$ ,  $0 \leq s \leq n$ ,  $0 \leq r \leq m$ , and  $f = (f_1, \dots, f_m) : \Omega \rightarrow \mathbb{C}^m$  be a bounded holomorphic mapping. Assume that  $W \in \Gamma_{(r,s)}^\infty$  such that  $\text{supp}(W) \subset \Omega$  and  $\text{supp}(W) \cap f^{-1}(0) = \emptyset$ .*

- i. *If  $\mathcal{T}_f W = 0$ , then there exists  $Y \in \Gamma_{(r+1,s)}^\infty$  such that*
  - a.  $\mathcal{T}_f Y = W$ ,
  - b.  $\text{supp}(Y) \subset \Omega$  and  $\text{supp}(Y) \cap f^{-1}(0) = \emptyset$ .
- ii. *If  $\mathcal{T}_f W = 0$  and  $\bar{\partial} W \in \Gamma_{(r,s+1)}^\infty$ , then there exists  $Y \in \Gamma_{(r+1,s)}^\infty$  such that*
  - a.  $\bar{\partial} Y \in \Gamma_{(r+1,s+1)}^\infty$  and  $\mathcal{T}_f Y = W$ ,
  - b.  $\text{supp}(Y) \subset \Omega$  and  $\text{supp}(Y) \cap f^{-1}(0) = \emptyset$ .

*Proof.* First let us prove the lemma in case  $r = m$ . In this case one can show that  $\mathcal{T}_f W = 0$  and  $\text{supp}(W) \cap f^{-1}(0) = \emptyset$  imply that  $W = 0$ . So we can choose  $Y = 0 \in \Gamma_{(m+1,s)}^\infty$ . For the rest of the proof we will assume that  $0 \leq r \leq m - 1$ .

Now let us prove i. Let  $\chi \in C_0^\infty(\Omega)$  be a smooth compactly supported cut-off function such that  $\chi = 1$  on a neighborhood of  $\text{supp}(W)$  and  $\text{supp}(\chi) \cap f^{-1}(0) = \emptyset$ . We define

$$g_j = \frac{\chi \bar{f}_j}{\sum_{l=1}^m |f_l|^2}$$

and

$$X = \sum_{j=1}^m e_j \otimes g_j \in \Gamma_{(1,0)}^\infty.$$

Then  $g_j \in C_0^\infty(\Omega)$  for  $j = 1, 2, \dots, m$  and  $\mathcal{T}_f X = 1 \in \Gamma_{(0,0)}^\infty$  on the support of  $W$  because  $\chi = 1$  on a neighborhood of  $\text{supp}(W)$  and  $\sum_{j=1}^m f_j(z) g_j(z) = 1$  whenever  $\chi(z) = 1$ .

Let us define  $Y = X \wedge W \in \Gamma_{(r+1,s)}^\infty$ . Then  $\text{supp}(Y)$  is a compact subset of  $\Omega$  and  $\text{supp}(Y) \cap f^{-1}(0) = \emptyset$ . Furthermore,  $\mathcal{T}_f X = 1$  on the support of  $W$  and

$$\mathcal{T}_f Y = \mathcal{T}_f(X) \wedge W - X \wedge \mathcal{T}_f W = 1 \wedge W = W$$

because  $\mathcal{T}_f W = 0$ .

To prove ii. we observe that, in the proof of i. above,  $X$  is smooth compactly supported in  $\Omega$ . Therefore, if  $\bar{\partial}W$  is bounded then so is  $\bar{\partial}Y$  as  $Y = X \wedge W$ .  $\square$

If  $f_j \in A^\infty(\Omega)$  for  $j = 1, 2, \dots, m$  in the lemma above, we have the following lemma.

**Lemma 2.** *Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$ ,  $V$  be an  $m$ -dimensional vector space, and  $f_j \in A^\infty(\Omega)$  for  $j = 1, 2, \dots, m$ . Assume that  $W \in \wedge^r V \otimes C_{(0,s)}^\infty(\bar{\Omega})$  for  $0 \leq r \leq m, 0 \leq s \leq n$ , and  $\text{supp}(W) \cap f^{-1}(0) = \emptyset$  where  $f = (f_1, \dots, f_m)$ . If  $\mathcal{T}_f W = 0$  then there exists  $Y \in \wedge^{r+1} V \otimes C_{(0,s)}^\infty(\bar{\Omega})$  such that  $\text{supp}(Y) \cap f^{-1}(0) = \emptyset$  and  $\mathcal{T}_f Y = W$ .*

*Proof.* The proof of this lemma is very similar to the proof of Lemma 1. The only difference is that we choose  $\chi \in C^\infty(\bar{\Omega})$  be a smooth function such that  $\chi = 1$  on a neighborhood of  $\text{supp}(W)$  and  $\text{supp}(\chi) \cap f^{-1}(0) = \emptyset$ .  $\square$

**Lemma 3.** *Let  $\Omega$  be a bounded  $L^\infty$ -pseudoconvex domain in  $\mathbb{C}^n$ ,  $f = (f_1, \dots, f_m) : \Omega \rightarrow \mathbb{C}^m$  be a bounded holomorphic mapping, and  $W \in \Gamma_{(r,s)}^\infty$  for  $0 \leq r \leq m$  and  $1 \leq s \leq n$  such that*

- i.  $\text{supp}(W) \subset \Omega$  and  $\text{supp}(W) \cap f^{-1}(0) = \emptyset$ ,
- ii.  $\bar{\partial}W = 0$  and  $\mathcal{T}_f W = 0$ .

*Then there exists  $Y \in \Gamma_{(r+1,s-1)}^\infty$  such that  $Y \in \text{Dom}_\infty(\bar{\partial})$  and  $\mathcal{T}_f \bar{\partial}Y = W$ .*

*Proof.* In case  $r = m$ , as in the proof of Lemma 1, one can show that if  $W$  satisfies the conditions of the lemma then  $W = 0$ . So we can choose  $Y = 0$ . For the rest of the proof we will assume that  $0 \leq r \leq m - 1$ .

First we will assume that  $\Omega$  is a bounded  $L^\infty$ -pseudoconvex domain. We will use a descending induction on  $s$  to prove this lemma. So let  $s = n, 0 \leq r \leq m - 1$ , and  $W \in \Gamma_{(r,n)}^\infty$  such that  $\text{supp}(W) \subset \Omega, \text{supp}(W) \cap f^{-1}(0) = \emptyset$ , and  $\mathcal{T}_f W = 0$  ( $\bar{\partial}W = 0$  as any  $(0, n)$ -form is  $\bar{\partial}$ -closed). Then i. in Lemma 1 implies that there exists  $Y_1 \in \Gamma_{(r+1,n)}^\infty$  with the following properties:

- i.  $\text{supp}(Y_1) \subset \Omega$  and  $\text{supp}(Y_1) \cap f^{-1}(0) = \emptyset$ ,
- ii.  $\mathcal{T}_f Y_1 = W$ .

Furthermore, since  $Y_1 \in \Gamma_{(r+1,n)}^\infty$  it is  $\bar{\partial}$ -closed. Then (since  $\Omega$  is  $L^\infty$ -pseudoconvex) there exists  $Y \in \Gamma_{(r+1,n-1)}^\infty$  such that  $\bar{\partial}Y = Y_1$ . That is,  $\mathcal{T}_f \bar{\partial}Y = W$ .

Now we will assume that the lemma is true for  $s = k + 1, k + 2, \dots, n$  and  $r = 0, 1, \dots, m - 1$ . Let  $0 \leq r \leq m - 1$  and assume that  $W \in \Gamma_{(r,k)}^\infty$  with the following properties:

- i.  $\text{supp}(W) \subset \Omega$  and  $\text{supp}(W) \cap f^{-1}(0) = \emptyset$ ,
- ii.  $\bar{\partial}W = 0$  and  $\mathcal{T}_f W = 0$ .

Then ii. in Lemma 1 implies that there exists  $Y_1 \in \Gamma_{(r+1,k)}^\infty$  such that

- i.  $\bar{\partial}Y_1 \in \Gamma_{(r+1,k+1)}^\infty$  and  $W = \mathcal{T}_f Y_1$ ,
- ii.  $\text{supp}(Y_1) \subset \Omega$  and  $\text{supp}(Y_1) \cap f^{-1}(0) = \emptyset$ .

Then

$$\mathcal{T}_f \bar{\partial} Y_1 = \bar{\partial} \mathcal{T}_f Y_1 = \bar{\partial} W = 0.$$

So  $\bar{\partial} Y_1$  satisfies the conditions in the lemma for  $s = k + 1$ . That is,  $\bar{\partial} Y_1 \in \Gamma_{(r+1, k+1)}^\infty$  such that

- i.  $\text{supp}(\bar{\partial} Y_1) \subset \Omega$  and  $\text{supp}(\bar{\partial} Y_1) \cap f^{-1}(0) = \emptyset$ ,
- ii.  $\bar{\partial} \bar{\partial} Y_1 = 0$  and  $\mathcal{T}_f \bar{\partial} Y_1 = \bar{\partial} W = 0$ .

By the induction hypothesis, there exists  $Y_2 \in \Gamma_{(r+2, k)}^\infty$  such that  $\bar{\partial} Y_2 \in \Gamma_{(r+2, k+1)}^\infty$  and  $\mathcal{T}_f \bar{\partial} Y_2 = \bar{\partial} Y_1$ . Then

$$\bar{\partial} \mathcal{T}_f Y_2 = \mathcal{T}_f \bar{\partial} Y_2 = \bar{\partial} Y_1.$$

We define  $Y_3 = Y_1 - \mathcal{T}_f Y_2 \in \Gamma_{(r+1, k)}^\infty$ . Then the equality above implies that

$$\mathcal{T}_f Y_3 = \mathcal{T}_f Y_1 - \mathcal{T}_f \mathcal{T}_f Y_2 = W$$

and  $\bar{\partial} Y_3 = \bar{\partial} Y_1 - \bar{\partial} \mathcal{T}_f Y_2 = 0$ . Since  $\Omega$  is  $L^\infty$ -pseudoconvex domain we conclude that there exists  $Y \in \Gamma_{(r+1, k-1)}^\infty$  such that  $\bar{\partial} Y = Y_3$ . That is,  $\mathcal{T}_f \bar{\partial} Y = W$ . Hence the proof of Lemma 3 is complete.  $\square$

**Lemma 4.** *Let  $\Omega$  be a smooth bounded pseudoconvex domain in  $\mathbb{C}^n$ ,  $V$  be an  $m$ -dimensional vector space, and  $f_i \in A^\infty(\Omega)$  for  $i = 1, \dots, m$ . Assume that  $W \in \wedge^r V \otimes C_{(0, s)}^\infty(\bar{\Omega})$  for  $0 \leq r \leq m$  and  $1 \leq s \leq n$  such that  $\text{supp}(W) \cap f^{-1}(0) = \emptyset$ ,  $\bar{\partial} W = 0$ , and  $\mathcal{T}_f W = 0$ . Then there exists  $Y \in \wedge^{r+1} V \otimes C_{(0, s-1)}^\infty(\bar{\Omega})$  such that  $\mathcal{T}_f \bar{\partial} Y = W$ .*

*Proof.* This proof is similar to the proof of Lemma 3 with the following changes: Instead of Lemma 1 we use Lemma 2 and, at the last step (since and  $f_j \in A^\infty(\Omega)$ ), we use the following result of Kohn [Koh73] (see also [CS01, Theorem 6.1.1]): Let  $\Omega$  be a smooth bounded pseudoconvex domain in  $\mathbb{C}^n$ ,  $1 \leq q \leq n$ , and  $u \in C_{(0, q)}^\infty(\bar{\Omega})$  with  $\bar{\partial} u = 0$ . Then there exists  $f \in C_{(0, q-1)}^\infty(\bar{\Omega})$  such that  $\bar{\partial} f = u$ .  $\square$

**Lemma 5.** *Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$  and  $f_j \in H^\infty(\Omega)$  for  $j = 1, \dots, m$  such that  $\sum_{j=1}^m |f_j|^2 > \varepsilon$  on  $\Omega$  for some  $\varepsilon > 0$  and  $\partial f_j \in L_{(1, 0)}^\infty(\Omega)$  for  $j = 1, \dots, m$ . Assume that  $W \in \Gamma_{(r, s)}^\infty$  for  $0 \leq r \leq m$  and  $0 \leq s \leq n$  such that  $\mathcal{T}_f W = 0$  and  $\bar{\partial} W \in \Gamma_{(r, s+1)}^\infty$ . Then there exists  $Y \in \Gamma_{(r+1, s)}^\infty$  such that  $\bar{\partial} Y \in \Gamma_{(r+1, s+1)}^\infty$  and  $\mathcal{T}_f Y = W$ .*

*Proof.* The proof will be similar to the proof of Lemma 1. Let  $V$  be a vector space of dimension  $m$  and  $\{e_1, e_2, \dots, e_m\}$  be a basis for  $V$ . We define

$$g_j = \frac{\bar{f}_j}{\sum_{l=1}^m |f_l|^2}$$

and  $X = \sum_{j=1}^m e_j \otimes g_j \in \Gamma_{(1, 0)}^\infty$ . Then  $g_j \in L^\infty(\Omega)$  and

$$\bar{\partial} g_j = \frac{\bar{\partial} \bar{f}_j}{\sum_{l=1}^m |f_l|^2} - \frac{\bar{f}_j \sum_{l=1}^m f_l \bar{\partial} f_l}{(\sum_{l=1}^m |f_l|^2)^2} \in L_{(0, 1)}^\infty(\Omega).$$

Furthermore,  $\bar{\partial}X = \sum_{j=1}^m e_j \otimes \bar{\partial}g_j \in \Gamma_{(1,1)}^\infty$ . Then  $Y = X \wedge W \in \Gamma_{(r+1,s)}^\infty$  satisfies the following properties:  $\bar{\partial}Y = \bar{\partial}X \wedge W + X \wedge \bar{\partial}W \in \Gamma_{(r+1,s+1)}^\infty$  and

$$\mathcal{T}_f Y = \mathcal{T}_f(X) \wedge W - X \wedge \mathcal{T}_f W = 1 \wedge W = W$$

as  $\mathcal{T}_f W = 0$ . □

**Proposition 1.** *Let  $\Omega$  be a bounded  $L^\infty$ -pseudoconvex domain in  $\mathbb{C}^n$  and  $f_j \in H^\infty(\Omega)$  for  $j = 1, \dots, m$  such that  $\sum_{j=1}^m |f_j|^2 > \varepsilon$  on  $\Omega$  for some  $\varepsilon > 0$  and  $\partial f_j \in L_{(1,0)}^\infty(\Omega)$  for  $j = 1, \dots, m$ . Assume that  $W \in \Gamma_{(r,s)}^\infty$  for  $0 \leq r \leq m$  and  $0 \leq s \leq n$  such that  $\bar{\partial}W = 0$  and  $\mathcal{T}_f W = 0$ . Then there exists  $Y \in \Gamma_{(r+1,s)}^\infty$  such that  $\bar{\partial}Y = 0$  and  $\mathcal{T}_f Y = W$ .*

*Proof.* We will use a descending induction on  $s$  as in the proof of Proposition 1. Let  $s = n$ . Any form of type  $(r, n)$  for  $0 \leq r \leq m$  is  $\bar{\partial}$ -closed. Then  $\bar{\partial}Y = 0$  and Lemma 5 implies that there exists  $Y \in \Gamma_{(r+1,n)}^\infty$  such that  $\mathcal{T}_f Y = W$ .

Now we will assume that the lemma is true for  $s = l+1, l+2, \dots, n$  and  $r = 0, 1, \dots, m$  to prove that it is also true for  $s = l \leq n-1$  and  $0 \leq r \leq m$ .

Assume that  $W \in \Gamma_{(r,l)}^\infty$  such that  $\bar{\partial}W = 0$  and  $\mathcal{T}_f W = 0$ . Then Lemma 5 implies that there exists  $\tilde{Y} \in \Gamma_{(r+1,l)}^\infty$  such that  $\bar{\partial}\tilde{Y} \in \Gamma_{(r+1,l+1)}^\infty$  and  $W = \mathcal{T}_f \tilde{Y}$ . Then

$$\mathcal{T}_f \bar{\partial}\tilde{Y} = \bar{\partial}\mathcal{T}_f \tilde{Y} = \bar{\partial}W = 0.$$

So  $\bar{\partial}\tilde{Y}$  satisfies the conditions in the lemma for  $s = l+1$ . That is,  $\bar{\partial}\tilde{Y} \in \Gamma_{(r+1,l+1)}^\infty$ ,  $\bar{\partial}\bar{\partial}\tilde{Y} = 0$  and  $\mathcal{T}_f \bar{\partial}\tilde{Y} = \bar{\partial}W = 0$ . Then, by the induction hypothesis, there exists  $Y_1 \in \Gamma_{(r+2,l+1)}^\infty$  such that  $\bar{\partial}Y_1 = 0$  and  $\mathcal{T}_f Y_1 = \bar{\partial}\tilde{Y}$ . Then since  $\Omega$  is a  $L^\infty$ -pseudoconvex domain there exists  $Y_2 \in \Gamma_{(r+2,l)}^\infty$  such that  $\bar{\partial}Y_2 = Y_1$ . Then

$$\bar{\partial}\mathcal{T}_f Y_2 = \mathcal{T}_f \bar{\partial}Y_2 = \mathcal{T}_f Y_1 = \bar{\partial}\tilde{Y}.$$

We define  $Y = \tilde{Y} - \mathcal{T}_f Y_2 \in \Gamma_{(r+1,l)}^\infty$ . Then the equality above implies that  $\bar{\partial}Y = \bar{\partial}\tilde{Y} - \bar{\partial}\mathcal{T}_f Y_2 = 0$  and

$$\mathcal{T}_f Y = \mathcal{T}_f \tilde{Y} - \mathcal{T}_f \mathcal{T}_f Y_2 = W.$$

Hence the proof of Proposition 1 is complete. □

As a corollary to the previous proposition (with  $W = 1$  and  $r = s = 0$ ) we get the following Corona type result. We refer the reader to [Kra14] and the references therein for more information about Corona problem on domains in  $\mathbb{C}^n$ .

**Corollary 3.** *Let  $\Omega$  be a bounded  $L^\infty$ -pseudoconvex domain in  $\mathbb{C}^n$  and  $f_j \in H^\infty(\Omega)$  for  $j = 1, \dots, m$  such that  $\sum_{j=1}^m |f_j|^2 > \varepsilon$  on  $\Omega$  for some  $\varepsilon > 0$  and  $\partial f_j \in L_{(1,0)}^\infty(\Omega)$  for  $j = 1, \dots, m$ . Then there exists  $g_i \in H^\infty(\Omega)$  for  $j = 1, \dots, m$  such that  $\sum_{j=1}^m f_j g_j = 1$ .*

## PROOFS OF RESULTS

The proofs of the theorems are mainly inspired by the proof in Bishop's paper [Bis89].

*Proofs of Theorems 1 and 2.* The proofs of both theorems are very similar. So we will present the proof of Theorem 1 and comment on how the proof of Theorem 2 differs as we go along.

Let  $\epsilon > 0$  and  $\lambda \in \mathbb{C}^m$ . Since  $g \in C(\overline{\Omega})$  and  $g|_{b\Omega \cup \Omega_f} = 0$ , there exist  $g^\lambda \in C^\infty(\overline{\Omega})$  such that

- i.  $\sup\{|g(z) - g^\lambda(z)| : z \in \overline{\Omega}\} < \epsilon$ ,
- ii  $\text{supp}(\bar{\partial}g^\lambda) \cap (b\Omega \cup f^{-1}(\lambda)) = \emptyset$ .

In the proof Theorem 2 the second condition above is replaced by  $\text{supp}(\bar{\partial}g^\lambda) \cap f^{-1}(\lambda) = \emptyset$ . This can be seen as follows: We choose an open set  $U_\epsilon$  in  $\mathbb{C}^n$  containing  $f^{-1}(\lambda)$  and  $g_\epsilon \in A^\infty(U_\epsilon \cap \Omega)$  such that  $|g - g_\epsilon| < \epsilon/2$  on  $f^{-1}(\lambda)$ . Then we choose  $\chi_\epsilon \in C_0^\infty(U_\epsilon)$  such that,  $0 \leq \chi_\epsilon \leq 1$ ,  $\chi_\epsilon = 1$  on a neighborhood of  $f^{-1}(\lambda)$ , and

$$\text{supp}(\chi_\epsilon) \cap \overline{\Omega} \subset \{z \in U_\epsilon \cap \overline{\Omega} : |g(z) - g_\epsilon(z)| < \epsilon\}.$$

Then we define  $g^\lambda = (1 - \chi_\epsilon)g + \chi_\epsilon g_\epsilon$ . Since  $g^\lambda$  is holomorphic on a neighborhood of  $f^{-1}(\lambda)$  we have  $\bar{\partial}g^\lambda = 0$  on the same neighborhood. Furthermore,  $|g^\lambda(z) - g(z)| = \chi_\epsilon(z)|g_\epsilon(z) - g(z)| < \epsilon$  for all  $z \in \overline{\Omega}$ .

Using Lemma 3 with  $r = 0, s = 1$ , and  $W = \bar{\partial}g^\lambda$  we get  $Y = \sum_{l=1}^m e_l \otimes H_l \in \Gamma_{(1,0)}^\infty$  such that

$$(1) \quad \bar{\partial}g^\lambda = \mathcal{T}_{f-\lambda} \bar{\partial}Y = \sum_{l=1}^m (f_l - \lambda_l) \bar{\partial}H_l^\lambda.$$

The above equality implies that

$$G_\lambda = g^\lambda - \sum_{l=1}^m (f_l - \lambda_l) H_l^\lambda$$

is a bounded holomorphic function.

In the proof of Theorem 2, we use Lemma 4 and get  $H_l^\lambda \in C^\infty(\overline{\Omega})$  for  $l = 1, \dots, m$  in the equation (1) and  $G_\lambda$  is smooth up to the boundary. Therefore, for  $z \in \Omega$  we have

$$|G_\lambda(z) - g^\lambda(z)| \leq \sum_{l=1}^m |f_l(z) - \lambda_l| \sum_{s=1}^m |H_s^\lambda(z)|.$$

Then the above inequality implies that for  $M_\lambda = \sum_{s=1}^m \|H_s^\lambda\|_{L^\infty(\Omega)} < \infty$  we have

$$(2) \quad |G_\lambda(z) - g^\lambda(z)| \leq M_\lambda |f(z) - \lambda|$$

for  $z \in \Omega$ .

Compactness of  $\overline{f(\Omega)}$  implies that we can choose a finite collection of points  $\{\lambda_j\}_{j=1}^k \subset \overline{f(\Omega)}$  such that  $\{B(\lambda_j, \epsilon M_{\lambda_j}^{-1})\}_{j=1}^k$  forms a finite open cover for  $\overline{f(\Omega)}$ . Let  $\{\chi_j\}_{j=1}^k$  be a smooth partition of unity on  $\overline{f(\Omega)}$  such that  $0 \leq \chi_j \leq 1$  and  $\text{supp}(\chi_j) \subset U_j$ . Then  $\{f^{-1}(B(\lambda_j, \epsilon M_{\lambda_j}^{-1}))\}_{j=1}^k$



is an cover for  $\Omega$  and  $|f(z) - \lambda^j| < \epsilon M_{\lambda^j}^{-1}$  for  $z \in f^{-1}(B(\lambda^j, \epsilon M_{\lambda^j}^{-1}))$ . Then for  $z \in \Omega$  we have

$$\begin{aligned} \left| \sum_{j=1}^k G_{\lambda^j}(z) \chi_j(f)(z) - g(z) \right| &\leq \sum_{j=1}^k |G_{\lambda^j}(z) - g(z)| \chi_j(f(z)) \\ &\leq \sum_{j=1}^k |G_{\lambda^j}(z) - g^{\lambda^j}| \chi_j(f(z)) + \sum_{j=1}^k |g^{\lambda^j}(z) - g(z)| \chi_j(f(z)) \\ &\leq \sum_{j=1}^k M_{\lambda^j} |f(z) - \lambda^j| \chi_j(f(z)) + \epsilon \sum_{j=1}^k \chi_j(f(z)) \\ &\leq 2\epsilon. \end{aligned}$$

Finally, the Stone-Weierstrass Theorem implies that  $\chi_j(f)$  can be approximated uniformly on  $\overline{\Omega}$  by elements of  $\mathbb{C}[f_1, \dots, f_m, \overline{f_1}, \dots, \overline{f_m}]$ . Hence the proofs of Theorems 1 and 2 are complete.  $\square$

Hartogs Extension Theorem together Theorem 2 lead to the following corollary.

**Corollary 4.** *Let  $\Omega$  be a bounded  $L^\infty$ -pseudoconvex domain in  $\mathbb{C}^n$ . Assume that  $f = (f_1, \dots, f_m) : \Omega \rightarrow \mathbb{C}^m$  be a bounded holomorphic mapping and  $g \in C(\overline{\Omega})$  such that  $\bar{\partial}g$  is supported away from  $b\Omega$  and the set of points at which the Jacobian of  $f$  has rank strictly less than  $n$ . Then  $g$  belongs to the closure of  $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$  in  $L^\infty(\Omega)$ .*

*Proof.* Since  $\bar{\partial}g$  vanishes near the boundary of  $\Omega$ , Hartogs Extension Theorem implies that there exists  $g_1 \in H^\infty(\Omega)$  such that  $g = g_1$  near the boundary of  $\Omega$ . Then  $g_2 = g - g_1 \in C(\overline{\Omega})$  and  $g_2$  is compactly supported in  $\Omega$ . Furthermore,  $g_2$  is holomorphic on a neighborhood of the set where the Jacobian of  $f$  has rank strictly less than  $n$ . Therefore, Theorem 2 implies that  $g_2$  can be approximated in the sup-norm by functions in  $H^\infty(\Omega)[\overline{f_1}, \dots, \overline{f_m}]$ . This completes the proof of the corollary.  $\square$

We finally end the paper with the proof of Corollary 2.

*Proof of Corollary 2.* We will use the fact that  $T_g$  can be defined by the following formula

$$\langle T_g \phi, \psi \rangle_{A^2(\Omega)} = \langle g \phi, \psi \rangle_{L^2(\Omega)}$$

for all  $\phi, \psi \in A^2(\Omega)$ . Since  $T_g$  commutes with  $T_{P(f)}$ , for any holomorphic polynomial  $P$ , we have

$$\langle g P(f), \psi \rangle = \langle T_g T_{P(f)}(1), \psi \rangle = \langle P(f) T_g(1), \psi \rangle$$

for all  $\psi \in A^2(\Omega)$ . Then  $\langle T_g(1) - g, \overline{P(f)} \psi \rangle = 0$  for all  $\psi \in A^2(\Omega)$ . Since, by Corollary 1, subspace generated by  $\{\overline{P(f)} \psi : \psi \in A^2(\Omega)\}$  is dense in  $L^2(\Omega)$ , we conclude that  $T_g(1) = g$ . That is,  $g$  is holomorphic.  $\square$

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*E-mail address:* Sonmez.Sahutoglu@utoledo.edu, Akaki.Tikaradze@utoledo.edu

UNIVERSITY OF TOLEDO, DEPARTMENT OF MATHEMATICS & STATISTICS, TOLEDO, OH 43606, USA